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CYCLES THROUGH SPECIFIED VERTICES

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Recently, various authors have obtained results about the existence of long cycles in graphs with a given minimum degree d. We extend these results to the case where only some of the vertices are known to have degree at least d, and we want to find a cycle through as many of these vertices as possible. If G is a graph on n vertices and W is a set of w vertices of degree at least d, we prove that there is a cycle through at least $\left\lceil \frac{w}{\lceil n/d\rceil - 1} \right\rceil$ vertices of W. We also find the extremal graphs for this property.

In this paper we consider the following question. Let G be a graph with n vertices, and let W be a set of w vertices of G, each with degree at least d. How large can s be such that we are guaranteed a cycle containing at least $s+1$ of the vertices of W? We obtain the best possible value in all cases.

Questions of this type have been studied for the case $W=V(G)$. In particular, for $w = n$ and $d = \lceil n/2\rceil$, the classical theorem of Dirac [3] tells us that G is Hamiltonian, so we can take $s=n-1$. More recently, Alon [1], Egawa and Miyamoto [4] and Bollobás and Häggkvist [2] answered the question for $w=n$ and all values of d, thus settling a question of Katchalski. Their results can be read out of our Theorem 1 on setting $w=n$. It is worth remarking that, although our result is considerably more general than those mentioned above, the proof we give is shorter and simpler than those in [1], [2] and [4].

We start with a few pieces of terminology. A graph G containing a distinguished subset W will be called *labelled by* W or simply *labelled*. For G a graph labelled by W and t an integer, we define a (t,W)*-cycle* to be a cycle in G containing at least t vertices of W.

The main aim of this paper is to prove the following result.

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Theorem 1. *Let G be a graph of order n, labelled by a set W of size $w \geq 3$, such that every vertex of W has degree at least $d \geq 1$. Set $s = \left\lceil \frac{w}{\lceil n/d \rceil - 1} \right\rceil - 1$ and suppose that $s \geq 2$, then there is an $(s+1, W)$-cycle.*

Let us first show that this result is best possible for all values of the parameters. Given n, w and d, we set $k = \lceil n/d \rceil$ and $s = \lceil w/(k-1) \rceil - 1$. We shall define a graph G with n vertices, labelled by a set W consisting of w vertices of G with degree at least d, so that G has no $(s+2, W)$-cycle.

If $s \leq d-1$, define G by taking $k-1$ disjoint copies of K_d, a separate vertex attached to all of them, and a set of $r = n - d(k-1) - 1 \geq 0$ isolated vertices. Choose W in such a way that it has at most $s+1$ vertices from each of the K_ds. Then each vertex of W has degree d and G has no $(s+2, W)$-cycle. On the other hand, if $s \geq d$, construct G as follows. To define the subgraph $H = G[W]$ spanned by W, take $k-1$ disjoint copies of K_s and a set Q of $w - (k-1)s$ additional vertices, so that $1 \leq |Q| \leq k-1$. To get H, join all vertices of each K_s to precisely one vertex of Q in such a way that each vertex of Q gets used at least once. The $n - w$ remaining vertices of G are taken as isolated vertices. Clearly every vertex of W has degree at least $s \geq d$, and there is no $(s+2, W)$-cycle in G.

There are several alternative ways of looking at Theorem 1: what it amounts to is that the largest number of W-vertices we can guarantee on a cycle is always of the form $\lceil w/l \rceil$ for some integer l, and so slight changes in n, w or d can produce large jumps in the value of s we obtain. Let us also restate Theorem 1 in the following cleaner but perhaps less appealing form.

Theorem 1′. *Let G be a graph on n vertices, labelled by a set W of w vertices of G, each of degree at least d. Suppose that $s \geq 2$, and that, for some integer $l \geq 2$, $w \geq s(l-1)+1$ and $n \leq dl$. Then G contains an $(s+1, W)$-cycle.*

Before going on to consider the proof of Theorem 1, let us see what we need to force a $(2, W)$-cycle. Here the conditions of the theorem are not quite sufficient, as shown by taking the w vertices of W to be a path, and adding other vertices of degree 1: $d-2$ attached to each interior vertex of the path, and $d-1$ attached to each endvertex. The resulting graph has just $(d-1)w+2$ vertices. (If our theorem extended to $s=1$, it would say that $n \leq dw$ was sufficient to force a $(2, W)$-cycle.) However, it is easy to see that this is the worst example.

Theorem 2. *Let G be a graph with n vertices, labelled by a set W of w vertices of degree at least d in G. If $n \leq (d-1)w+1$, then G has a $(2, W)$-cycle.*

Proof. If G contains no $(2, W)$-cycle, then without loss of generality the graph G is a tree, since we can remove any edges not incident with W. But a tree with w vertices of degree at least d has at least $(d-1)w+2$ vertices. ∎

Let us now move on to the proof of the main result.

Proof of Theorem 1′. Clearly we may assume that $w = s(l-1)+1$ and $n = dl$, for some $l \geq 2$. We may also assume that every edge of G is incident with a vertex of W.

The proof is by induction on l. Let us first prove the result for $l = 2$. In this case, the proof is essentially the same as that of Dirac's Theorem [3]. Suppose then that $w = s+1$, $n = 2d$, and that there is no cycle through all the vertices of W.

If there is no path in G containing all the W-vertices, we can add any edge between W-vertices to G without creating an $(s+1,W)$-cycle. Thus we may assume that there is such a path. Take any path P in G, with endpoints in W, through all $s+1$ vertices of W.

The endvertices w_1 and w_2 of P clearly have no common neighbours off P. So they send at most $n-|P|=2d-|P|$ edges out of P. But if z is the successor of y on P, then at most one of the edges w_1z and w_2y is in G, since if both are in G then $w_1\dots yw_2\dots zw_1$ is a cycle containing all vertices of P. Thus there are at most $|P|-1$ edges from w_1 and w_2 to the rest of P. Hence $d(w_1)+d(w_2)<2d$, a contradiction.

Now suppose that $l\geq 3$, $w=s(l-1)+1$, $n=dl$, that there is no $(s+1,W)$-cycle, and that the result is true for smaller values of l. Let P be a path with endpoints in W containing the maximum number of W-vertices. As above, we may assume that P contains at least $s+1$ vertices of W. Let w_k be the kth W-vertex on P, for $k=1,\dots,t\equiv|W\cap P|\geq s+1$.

The aim is to remove a set D of at least d vertices from G such that the new graph $G'=G-D$ contains a subset W' of W with $w-s$ vertices, all of whose neighbours belong to G'. Then we apply the induction hypothesis to the graph G' labelled by W'.

The sets D and W' are defined as follows:

$$\begin{aligned}
D_1 &= \Gamma(w_1) - V(P),\\
D_2 &= \{u \in V(P) : \text{ the successor of } u \text{ on } P \text{ is a neighbour of } w_1\},\\
D &= D_1 \cup D_2,\\
W' &= W - \{w_1,\dots,w_s\}.
\end{aligned}$$

These definitions are illustrated in Figure 1.

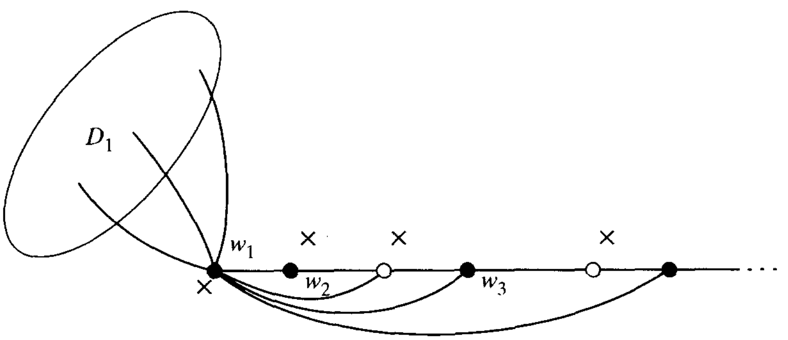

Fig. 1. The path P and the set $D=D_1\cup D_2$; the vertices of W are marked with solid circles, and the vertices of D_2 with crosses

Clearly $|D|\geq d$ and $|W'|=w-s$. We claim that (i) W' and D are disjoint and (ii) for $w'\in W'$, all neighbours of w' belong to $G'=G-D$, so in particular $d_{G'}(w')=d_G(w')\geq d$.

Indeed, suppose $w' \in W'$. By maximality of P, $w' \notin D_1$. If $w' \in D_2$, and v is its successor on P, then $w_1 \ldots w'vw_1$ is an $(s+1, W)$-cycle in G, contrary to hypothesis. Thus $w' \notin D$, proving (i).

For (ii), let x be a neighbour of w'. If $x \in D_1$, then either $w' \notin P$, when $w'xw_1 \ldots w_t$ is a path containing more W-vertices than P, or $w' \in P$, in which case $w'xw_1 \ldots w'$ is an $(s+1, W)$-cycle – in both cases a contradiction. Now suppose that $x \in D_2$ and v is its successor on P. If $w' \notin P$, then $w'x \ldots w_1v \ldots w_t$ is a path contradicting the maximality of P. On the other hand, if $w' \in P$, then it must occur after v, since otherwise $w_1 \ldots vw_1$ is an $(s+1, W)$-cycle, and therefore $w'x \ldots w_1v \ldots w'$ is an $(s+1, W)$-cycle, a contradiction. Thus $x \notin D$, establishing (ii).

The graph G' has at most $n-d = (l-1)d$ vertices, and each vertex of W' has degree at least d in G'. There are $w - s = (l-2)s+1$ vertices in W', so by the induction hypothesis there is an $(s+1, W')$-cycle in G'. This cycle is also an $(s+1, W)$-cycle in G, contradicting our assumptions.

This completes the proof. ∎

As is almost always the case, we can read out of our proof an Ore-type extension of the result, i.e., we can replace our condition on the minimum degree of a W-vertex by a condition on the sum of degrees of each pair of non-adjacent W-vertices. Suppose then that $d(w_1)+d(w_2) \geq c$ whenever w_1 and w_2 are non-adjacent vertices of W, and let us see what our proof tells us in this case. For the case $l=2$, taking w_1 and w_2 to be the endvertices of our path, we see that if G satisfies the degree conditions and has no cycle through all $s+1$ vertices of W, then G has at least $c+1$ vertices. In the induction step, once we have our long path, we can take either endpoint as w_1: in particular we can take one with degree at least $\lceil c/2 \rceil$. Then, proceeding as before, we can reduce w by s and n by at least $\lceil c/2 \rceil$. So we have the following result, again extending a result of Egawa and Miyamoto [4].

Theorem 3. *Let G be a graph on n vertices, labelled by a set W of w vertices of G such that each pair w_1, w_2 of non-adjacent vertices of W satisfies $d(w_1)+d(w_2) \geq c$. If $c=2d$ is even, set $k = \left\lceil \frac{n}{d} \right\rceil$: if $c = 2d+1$ is odd, set $k = \left\lceil \frac{n+1}{d+1} \right\rceil$. Now set $s = \left\lceil \frac{w}{k-1} \right\rceil - 1$. If $s \geq 2$, then there is an $(s+1, W)$-cycle.* ∎

Theorem 3 is again best possible. Extremal graphs can be obtained as for Theorem 1, except that if $c=2d+1$ is odd and $s \leq d-1$, an extremal graph is given by taking $k-1$ copies of K_{d+1} and one K_d, all disjoint, each with at most $s+1$ vertices of G, and adding one other vertex joined to every other.

Finally, we consider the problem of determining the set of extremal graphs for Theorem 1′. If W is a set of $w \geq (k-1)s+1$ vertices of a graph G, each of degree at least d, and G contains no $(s+1, W)$-cycle, then Theorem 1 tells us that G must have at least $kd+1$ vertices. Which are the graphs with exactly $kd+1$ vertices? We have been able to solve this problem completely. Our solution follows the proof of Theorem 1′ but, perhaps surprisingly, there is a fair amount of extra work to be done.

Note that what we are seeking is not quite the narrowest possible class of extremal graphs for the property of having no $(s+1, W)$-cycle: for each fixed k, s

and d, we could insist that $w = ks$ and $n = kd + 1$. Here we allow w to be rather less than ks. Alternative approaches allow n to be greater than its minimum value $kd+1$, resulting in several extra vertices, which can be included into the graph in a wide variety of ways. So this seems to be the most general extremal problem which can readily be tackled.

Suppose we are given parameters d and $s \geq 2$. We define two types of subgraph of a labelled graph. It will turn out that our extremal labelled graphs are made up from subgraphs of these types.

An *A-graph* A is a subgraph of a labelled graph induced by a set of $d+1$ vertices, containing at least one vertex of W connected to all other vertices of A, and at most s vertices of W of degree greater than 1 in A.

A *B-graph* is also an induced subgraph of a labelled graph. Its vertices fall into two classes Q and R; every vertex in Q is adjacent to every vertex in R, and there are no edges inside R. Class Q is of order d; class R is of order $ld+1$ for some integer $l \geq 1$, and is contained in W.

Theorem 4. *Suppose G is a graph on $n = kd+1$ vertices for some integer k, labelled by a set W of $w \geq (k-1)s+1$ vertices, each of degree at least d, where $s \geq 2$ is such that there is no $(s+1, W)$-cycle. Then G is a connected graph which is the union of some A-graphs and at most one B-graph, called the units of G, such that the intersection of any two units has at most one vertex, which is a cutvertex of G.*

Furthermore, one of the following three cases occurs. (See Figure 2.)

(i) *All the units are A-graphs, and every vertex of W has degree at least d.*

(ii) *There is a B-graph, $s = d$, $w = (k-1)s+1$, the class Q of the B-graph consists of all the d vertices of G not in W, and each A-graph consists of a clique containing one vertex of Q and d vertices of W.*

(iii) *$s > d$, $n = 2d+1$, $w = s+1$, and G consists of a single B-graph with $l = 1$.*

Conversely, if a labelled graph G with n, d, s and w as above can be decomposed into units satisfying (i), (ii) or (iii), then it has no $(s+1, W)$-cycle.

Proof. We remark that the units of G may be thought of as blocks of the graph; however in case (i) it is possible, as shown in Figure 2(i), for one of the A-graphs not to be 2-connected.

Let G be a graph labelled by W, with n, w, k, d and s as in the theorem, and suppose that G has no $(s+1, W)$-cycle.

Suppose first that G has a subgraph B_0 which is a B-graph. Then there is a cycle inside B_0 including d vertices of $R \subseteq W$, so we have $s \geq d$. Furthermore, any cycle of length $2d$ in B_0 includes at least $2d - (n - w)$ vertices of W, so we have $s \geq 2d - n + w \geq 2d - (kd+1) + ((k-1)s+1)$ and therefore $(k-2)d \geq (k-2)s$. Thus we find that, if G contains a B-graph as a subgraph, then $w = (k-1)s+1$, all the vertices of G not in W are in Q, and either $s = d$, or $s > d$ and $k = 2$.

This certainly implies that, if G is a union of A-graphs and B-graphs as in the theorem, then G is of one of the three forms listed. Indeed, if G has a B-subgraph and we have $k = 2$ and $s > d$, we are in case (iii); if G has a B-subgraph and $s = d$, then each of the A-subgraphs contain a vertex not in W, which is therefore in Q, as in case (ii); finally if G has no B-subgraphs we have case (i).

The main part of the proof will follow that of Theorem 1′. Observe that it is sufficient to prove the result for $w = (k-1)s+1$ since, if w is larger, we can unlabel some of the vertices of W to obtain this value of w. We have already seen that a

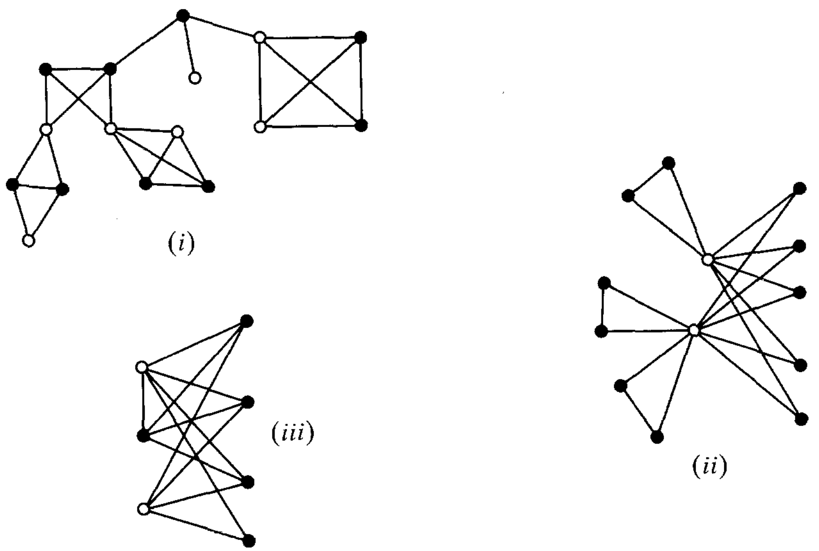

Fig. 2. The three cases in Theorem 4. Vertices of W are indicated by solid circles.
(i) $d=3$, $s=2$, $k=5$, $n=16$, $w=9$, *and there are* 5 *units, all A-graphs;*
(ii) $d=2$, $s=2$, $k=6$, $n=13$, $w=11$, *and there is one B-graph and* 3 *A-graphs;*
(iii) $d=3$, $s=4$, $k=2$, $n=7$, $w=5$, *and the graph consists of a single B-graph*

graph of type (ii) or (iii) above cannot have $w>(k-1)s+1$, so the resulting graph must be of type (i).

In the course of the proof we shall make use of appropriate long paths. Call a path in G *maximal* if it has the maximum possible number of vertices of W and, subject to this, fewest non-W-vertices.

Let us first consider the case where $k=2$. So we have $n=2d+1$ vertices in G, $w=s+1$, and no cycle going through all of the W-vertices. It is easily checked, using Theorem 1, that G is connected.

Let P be a maximal path in G. If we have a cycle through all the vertices of P, then we can construct either an $(s+1,W)$-cycle or a path containing more vertices of W, in both cases a contradiction, so there is no such cycle in G.

The two endpoints of P are both in W: call them w_1 and w_2. They are not adjacent, and they have no common neighbour outside P. So the number of edges from $\{w_1,w_2\}$ to vertices not on P is at most $n-|P|$. Let Γ denote the set of neighbours of w_2 on P, and Γ^- denote the set of predecessors of neighbours of w_1 on P. Then as usual $\Gamma\cap\Gamma^-$ is empty, and neither set contains w_2, so $|\Gamma|+|\Gamma^-|\leq |P|-1$. Hence $d(w_1)+d(w_2)\leq n-1=2d$. By the degree conditions on w_1 and w_2 we have equality everywhere. Thus, for any maximal path P, $P-w_2$ is the disjoint union of Γ and Γ^-, and every vertex outside P is adjacent to exactly one of w_1 and w_2. Note that $w_1\in\Gamma^-$, and the predecessor x of w_2 is in Γ.

Claim. *Let P be a maximal path. Either the elements of Γ and Γ^- alternate on P, or all of Γ^- precedes all of Γ on P.*

To prove this claim, let P be a maximal path on which we do not have all of Γ^- preceding all of Γ. Then somewhere along the path is an element a of Γ followed immediately by an element b of Γ^-. Consider the rotated path $P' = w_1 \ldots a w_2 x \ldots b$. The new endvertex b is in W, since P is maximal, and clearly P' is also a maximal path. Thus $P' - b$ is the disjoint union of the set of neighbours of b and the set of predecessors of neighbours of w_1 (on P'). Since b is not adjacent to w_2, we see that x is adjacent to w_1. Hence the predecessor y of x on P is in Γ^-.

Now suppose in addition that there are two consecutive elements u and v of Γ on P. These must precede y on P, so $w_1 \ldots u w_2 v \ldots x w_1$ is a cycle picking up all of P.

Thus, unless all of Γ^- precedes all of Γ, we do not have any two consecutive elements of Γ on P. By the same argument we do not have two consecutive elements of Γ^- either, so the elements alternate, proving the claim.

We now consider two possibilities.

(a) There is a maximal path P on which all of Γ^- precedes all of Γ.

Let z be the first vertex of P in Γ. Thus w_1 is adjacent to every vertex up to and including z on P, and w_2 is adjacent to all vertices from z onwards. By rotating the path we see that every vertex of P except possibly z is in W, and that the set of vertices of P up to and including z on P forms a clique in G, as does the set of vertices of P from z onwards.

If there is any path between these two cliques avoiding z then we can use this path to form a cycle including all of P, so z disconnects G. By the degree conditions on w_1 and w_2, we see that each component of $G - z$ must have order d, so G consists of two A-graphs, as in case (i).

(b) On every maximal path, the elements of Γ and Γ^- alternate.

Take any maximal path P, and let y be any vertex of P in $\Gamma^- - w_1$. Let v be the predecessor of y on P, and u the predecessor of v. Since $u \in \Gamma^-$, v is adjacent to w_2 and we can form a rotated path $P' = w_1 \ldots u v w_2 \ldots y$ with endvertices w_1 and y. Both P and P' are maximal, hence every vertex not in $V(P) = V(P')$ is adjacent to exactly one of w_1 and w_2, and to exactly one of w_1 and y. Therefore the set of neighbours of y not on P is equal to the set of neighbours of w_2 not on P. But we can also form a rotated path with endvertices y and w_2, so the set of neighbours of y outside $V(P)$ is also equal to the set of neighbours of w_1 outside $V(P)$. Hence this set is empty, and all the neighbours of w_1 and w_2 are on P.

Since w_2 is adjacent to alternate vertices on P, all $2d+1$ vertices of G are in fact on P. Let Q be the set consisting of the d neighbours of w_1, which is the same as the set of neighbours of w_2. Every vertex z of $R = V(G) - Q$ can be rotated to both ends of the path, so the maximality of P implies that z belongs to W, and also, using the preceding argument, that z is adjacent to all the vertices of Q. Thus G contains the complete bipartite graph with classes Q and R. Since $|Q| = d$ and $|R| = d+1$ and G is not Hamiltonian, there is no R–R edge. Thus G consists of one B-graph, and we have case (ii) or (iii), depending on the value of s.

This completes the proof in the case $k = 2$.

Now we turn to the general case. Thus we suppose $k \geq 3$, the result is true for all smaller k, and we have a labelled graph G satisfying the conditions with no $(s+1, W)$-cycle. Let P be a maximal path in G. We proceed as in the proof of Theorem 1$'$. Define a subset W' of W by removing the first s vertices $w_1, \ldots w_s$ of

W on P; let D_1 be the set of neighbours of w_1 outside P, and let D_2 be the set of predecessors of neighbours of w_1 on P. Then form a graph G' by deleting the vertices of $D \equiv D_1 \cup D_2$ from G. (Figure 1 earlier illustrates these definitions.)

As in the proof of Theorem 1′, the vertices of W' have degree at least d in G', and so the graph G' labelled by W' satisfies the conditions of our present theorem for $k-1$. Hence $|D|=d$, and by the induction hypothesis G' is made up of A-graphs and B-graphs. By the remarks at the beginning of the proof, if $s>d$ then G cannot contain a B-graph and so neither can G'. Therefore G' is constructed as in cases (i) or (ii).

Let $A_1, A_2, \ldots, A_t$ be the units in a decomposition of G' into A-graphs and B-graphs. If in G there is a path between vertices x and y of G' which includes no other vertices of G', then x and y must be in the same A_i, since otherwise the graph G^* formed by adding the edge xy to G' would contain no $(s+1, W)$-cycle, contradicting the induction hypothesis.

We now distinguish two cases, depending on whether G' is composed as in case (i) or case (ii) of the theorem.

Let us first consider the case when G' is as in case (ii). Suppose then that A_1 is a B-graph with classes Q and R. In this case, $s=d$ and all the vertices of G not in W are in Q. So all the d vertices of D are in W. Suppose some element z of D has at most one neighbour outside D. Then, since $z \in W$ and so $d(z) \geq d$, z must be adjacent to every other vertex of D as well as a vertex y outside D. If any other vertex w of D is adjacent to a vertex x other than y outside D, then the path $xwzy$ can be extended to an $(s+1, W)$-cycle in G, a contradiction. Thus in this case the vertices of D together with y span a clique. Hence y is not in W, so must be in Q. Therefore these vertices form an A-graph, and G is as in (ii).

Thus we may assume that every vertex of D is joined to at least two vertices of G'. The only way that this can fail to give an $(s+1, W)$-cycle is if all neighbours of D in G' are in Q. If any two vertices of D are adjacent, this again gives an $(s+1, W)$-cycle in G, so there are no edges inside D. The degree conditions now imply that every vertex of D is adjacent to every vertex of Q. The classes Q and $R' = R \cup D$ then span a larger B-graph, and G is again as in case (ii) of the theorem.

Now we turn to the case where G' is as in (i), that is, made up entirely of A-graphs. For $i=1, \ldots, t$, let A'_i be the set of vertices w of G such that there is a path between two distinct vertices x and y of A_i, passing through w. Thus $A_i \subseteq A'_i$, and no vertex of G' not in A_i is in A'_i. Note also that no vertex of D is in more than one A'_i, since that would imply the existence of a path through D between vertices of G' which are not in the same A_i. We shall call such a path *forbidden* – we have seen that there are none in G.

Suppose that some vertex w of W is in no A'_i. Then w is in D, and is adjacent to at most one vertex y of G', and so is adjacent to all other vertices of D. Then also no other vertex of D is adjacent to a vertex in G', other than y, so the set $D \cup \{y\}$ of vertices of G spans an A-graph A_{t+1}, and G is the union of $A_1, \ldots, A_{t+1}$ as required.

So we may assume that every vertex of W is in some A'_i. Let A'_j be the unit containing the vertex w_1 which is the initial vertex of P. As a next step, we shall prove that the only vertices of G' adjacent to a vertex of D are in A'_j, and so $A'_i = A_i$ if $i \neq j$.

Suppose then that a vertex y of $G' - A_j$ is adjacent to a vertex x of D. Say $y \in A_i$. Since w_1 is in A'_j, there is a path P' from w_1 to a vertex z of $A_j - A_i$ not including any other vertex of G'. If $x \in D_1$, then x is adjacent to w_1, giving us a forbidden path between y and z, a contradiction. Therefore $x \in D_2$, and so w_1 is adjacent to the successor v of x on P. If now $v \in D$, then $yxvw_1P'z$ is again a forbidden path between y and z; if $v \in G' - A_j$, then $vw_1P'z$ is a forbidden path; and if $v \in G' - A_i$, then yxv is a forbidden path. So v is the unique vertex in $A_i \cap A_j$. Now the walk $yxPw_1P'z$, containing the section of P between x and w_1, does not include v, and so includes a forbidden path between A_i and A_j, which is once more a contradiction. Hence there are no edges between D and $G' - A_j$, as desired.

Therefore each of the A'_i other than A'_j contains exactly $d+1$ vertices. Also, every vertex of W' not in A'_j has at least d neighbours in G'.

Since $k \geq 3$, G' has at least two A-graphs. Let A_l be an A-graph other than A_j which has just one vertex y in common with any other A-graph in G'. (This A_l can be thought of as an endblock in G'.) Every vertex of W in A_l except possibly y has d neighbours in A_l, and so is joined to all other vertices of A_l. Thus if there are at least 2 vertices of W in $A_l - y$, then there is a cycle in A_l through all the vertices of $W \cap A_l$; therefore in any case there are at most s vertices of W in A_l.

Finally, the labelled graph G^* obtained by deleting $A_l - y$ from G, and removing y from W if necessary, satisfies the conditions of the theorem for $k-1$. So by the induction hypothesis G^* is as in cases (i) or (ii) of the theorem, and hence so is G.

Conversely it is obvious that, if a labelled graph G can be decomposed into units as described by any of (i), (ii) or (iii), then G has no $(s+1, W)$-cycle, and every vertex of W has degree at least d.

This completes the proof. ∎

References

[1] N. Alon: The largest cycle of a graph with a large minimal degree, *J. Graph Theory* **10** (1986), 123–7.

[2] B. Bollobás, and R. Häggkvist: The circumference of a graph with given minimal degree, *A Tribute to Paul Erdős* (A. Baker, B. Bollobás and A. Hajnal eds.), Cambridge University Press 1989.

[3] G. A. Dirac: Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.

[4] Y. Egawa, and T. Miyamoto: The longest cycles in a graph G with minimum degree at least $|G|/k$, *J. Combinatorial Theory, Series B* **46** (1989), 356–362.

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